

# Spectral radius and traceability of connected claw-free graphs

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## Abstract

Let  $G$  be a connected claw-free graph on  $n$  vertices and  $\overline{G}$  be its complement graph. Let  $\mu(G)$  be the spectral radius of  $G$ . Denote by  $N_{n-3,3}$  the graph consisting of  $K_{n-3}$  and three disjoint pendent edges. In this note we prove that:

- (1) If  $\mu(G) \geq n - 4$ , then  $G$  is traceable unless  $G = N_{n-3,3}$ .
- (2) If  $\mu(\overline{G}) \leq \mu(\overline{N_{n-3,3}})$  and  $n \geq 24$ , then  $G$  is traceable unless  $G = N_{n-3,3}$ .

Our works are counterparts on claw-free graphs of previous theorems due to Lu et al., and Fiedler and Nikiforov, respectively.

**Keywords:** Spectral radius; Traceability; Claw-free graph

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## 1 Introduction

Let  $G$  be a graph. The *eigenvalues* of  $G$  are the eigenvalues of the adjacency matrix of  $G$ . Since the adjacency matrix of  $G$  is real and symmetric, all its eigenvalues are real. The *spectral radius* of  $G$ , denoted by  $\mu(G)$ , is the spectral radius of its adjacency matrix, i.e., the maximum among the absolute values of its eigenvalues. By Perron-Frobenius' theorem (see Theorem 0.3 of [4]),  $\mu(G)$  is equal to the largest eigenvalue of  $G$ .

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Let  $G$  be a graph. We use  $e(G)$  to denote *the number of edges* of  $G$ . Let  $S \subset V(G)$ . We use  $G[S]$  to denote the subgraph of  $G$  induced by  $S$  and  $G - S$  to denote the subgraph of  $G$  induced by  $V(G) \setminus S$ . For a subgraph  $H$  of  $G$ , we use  $G - H$  instead of  $G - V(H)$ . For two subgraphs  $H, H'$  of  $G$ , we use  $e_G(H, H')$  (or shortly,  $e(H, H')$ ) to denote the number of edges with one vertex in  $H$  and the other one in  $H'$ .

By  $\overline{G}$  we denote the *complement* of  $G$ . Let  $G_1$  and  $G_2$  be two graphs. We denote by  $G_1 + G_2$  the *disjoint union* of  $G_1$  and  $G_2$ , and by  $G_1 \vee G_2$  the *join* of  $G_1$  and  $G_2$ .

A graph  $G$  is *traceable* if it has a Hamilton path, i.e., a path containing all vertices of  $G$ ; and  $G$  is *Hamiltonian* if it has a Hamilton cycle, i.e., a cycle containing all vertices of  $G$ . Note that every Hamiltonian graph is traceable. Hamiltonian properties of graphs have received much attention from graph theorists. A fundamental theorem due to Dirac [5] states that every graph on  $n$  vertices is traceable if the degree of every vertex is at least  $(n - 1)/2$ . Up to now, there also has been some references on the spectral conditions for Hamilton paths or cycles. We refer the reader to [3, 8, 10, 15, 17, 19].

In particular, Fiedler and Nikiforov [8] gave tight sufficient conditions for the existence of a Hamilton path in terms of the spectral radii of a graph and its complement.

**Theorem 1** (Fiedler and Nikiforov [8]). *Let  $G$  be a graph on  $n$  vertices. If  $\mu(G) \geq n - 2$ , then  $G$  is traceable unless  $G = K_{n-1} + K_1$ .*

**Theorem 2** (Fiedler and Nikiforov [8]). *Let  $G$  be a graph on  $n$  vertices. If  $\mu(\overline{G}) \leq \sqrt{n - 1}$ , then  $G$  is traceable unless  $G = K_{n-1} + K_1$ .*

**Remark 1.** Note that  $\mu(K_{n-1} + K_1) = \mu(K_{n-1}) = n - 2$  and  $\mu(\overline{K_{n-1} + K_1}) = \mu(K_{1, n-1}) = \sqrt{n - 1}$ .

Since the connectedness is necessary for studying traceability of graphs. Lu, Liu and Tian [15] presented a sufficient condition for a connected graph to be traceable.

**Theorem 3** (Lu, Liu and Tian [15]). *Let  $G$  be a connected graph of order  $n \geq 7$ . If  $\mu(G) \geq \sqrt{(n - 3)^2 + 3}$ , then  $G$  is traceable.*

Lu et al.'s lower bound of spectral radius was sharpened in [17].

**Theorem 4** (Ning and Ge [17]). *Let  $G$  be a connected graph on  $n \geq 7$  vertices. If  $\mu(G) \geq n - 3$ , then  $G$  is traceable unless  $G = K_1 \vee (K_{n-3} + 2K_1)$ .*

The bipartite graph  $K_{1,3}$  is called a *claw*. A graph is called *claw-free* if it contains no induced subgraph isomorphic to  $K_{1,3}$ . Claw-free graphs have been a very popular field of study, not only in the context of Hamiltonian properties. One reason is that the very

natural class of line graphs turns out to be a subclass of the class of claw-free graphs. However, not every claw-free graph is Hamiltonian. There are examples of 3-connected non-Hamiltonian claw-free (even line) graphs, but it is a long-standing conjecture that all 4-connected claw-free graphs are Hamiltonian (and then, traceable). It is interesting to note that the lower bound on the degrees in Dirac's theorem for traceability was lowered to  $(n-2)/3$  by Matthews and Sumner [16] for claw-free graphs. For a survey on claw-free graphs, we refer the reader to Faudree et al. [7].

Motivated by the relationship between Dirac's theorem and Matthews-Sumner's theorem, in this note we will improve the lower bound in Theorem 3 and give an analogue of Theorem 2 for connected claw-free graphs.

Our main results will be listed as follows. By  $N_{n-3,3}$  we denote the graph consisting of a complete graph  $K_{n-3}$  with three disjoint pendent edges.

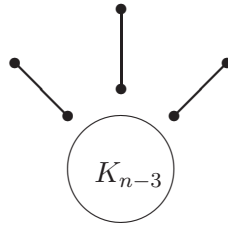


Fig. 1. Graph  $N_{n-3,3}$ .

**Theorem 5.** *Let  $G$  be a connected claw-free graph on  $n$  vertices. If  $\mu(G) \geq n-4$ , then  $G$  is traceable unless  $G = N_{n-3,3}$ .*

**Theorem 6.** *Let  $G$  be a connected claw-free graph on  $n \geq 24$  vertices. If  $\mu(\overline{G}) \leq \mu(\overline{N_{n-3,3}})$ , then  $G$  is traceable unless  $G = N_{n-3,3}$ .*

## 2 Preliminaries

In this section, we first extend the concept of claw-free graphs to a general one. Let  $R$  be a given graph. The graph  $G$  is called  $R$ -free if  $G$  contains no induced subgraph isomorphic to  $R$ . We will also use three special graphs  $L$ ,  $M$  and  $N$  (see Fig. 2). Note that  $N = N_{3,3}$ .

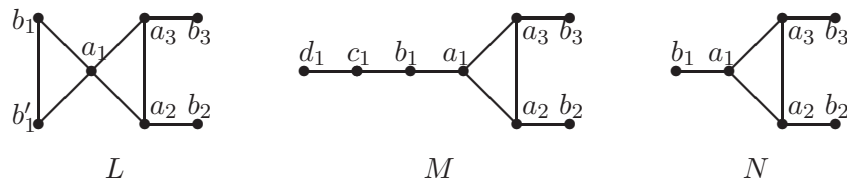


Fig. 2. Graphs  $L$ ,  $M$  and  $N$ .

The following two theorems concerning traceability of claw-free graphs are used in our proofs.

**Theorem 7** (Duffus, Gould and Jacobson [6]). *Every connected claw-free and  $N$ -free graph is traceable.*

Adopting the terminology of [9], we say that a graph is a *block-chain* if it is nonseparable or it has connectivity 1 and has exactly two end-blocks.

**Theorem 8** (Li, Broersma and Zhang [13]). *Let  $G$  be a block-chain. If  $G$  is claw-free and  $M$ -free, then  $G$  is traceable.*

One important tool for studying Hamiltonian properties of claw-free graphs is the closure theory introduced by Ryjáček [18]. It is also useful for our proof. To ensure the completeness of our text, we include all the terminology and notations as follows. For other more information, see [18].

Let  $G$  be a graph. Following [18], for a vertex  $x \in V(G)$ , if the neighborhood of  $x$  induces a connected but non-complete subgraph of  $G$ , then we say that  $x$  is *eligible* in  $G$ . Set  $B_G(x) = \{uv : u, v \in N(x), uv \notin E(G)\}$ . The graph  $G'_x$ , constructed by  $V(G'_x) = V(G)$  and  $E(G'_x) = E(G) \cup B_G(x)$ , is called the *local completion of  $G$  at  $x$* .

As shown in [18], the *closure* of a claw-free graph  $G$ , denoted by  $cl(G)$ , is defined by a sequence of graphs  $G_1, G_2, \dots, G_t$ , and vertices  $x_1, x_2, \dots, x_{t-1}$  such that

- (1)  $G_1 = G$ ,  $G_t = cl(G)$ ;
- (2)  $x_i$  is an eligible vertex of  $G_i$ ,  $G_{i+1} = (G_i)'_{x_i}$ ,  $1 \leq i \leq t-1$ ; and
- (3)  $cl(G)$  has no eligible vertices.

**Theorem 9** (Ryjáček [18]). *Let  $G$  be a claw-free graph. Then  $cl(G)$  is also claw-free.*

**Theorem 10** (Brandt, Favaron and Ryjáček [1]). *Let  $G$  be a claw-free graph. Then  $G$  is traceable if and only if  $cl(G)$  is traceable.*

A claw-free graph  $G$  is said to be *closed* if  $cl(G) = G$ . It is not difficult to see that for every vertex  $x$  of a closed graph  $G$ ,  $N_G(x)$  is either a clique, or the disjoint union of two cliques in  $G$  (see [18]). In the following, we say a vertex  $x$  of a graph  $G$  is a *bad vertex* of  $G$  if  $N_G(x)$  is neither a clique, nor the disjoint union of two cliques. So every closed graph has no bad vertices.

**Lemma 1.** *Let  $G$  be a closed claw-free graph. If there are two nonadjacent vertices of  $G$  have degree sum at least  $n-1$ , then  $G$  is traceable.*

*Proof.* Let  $x, y$  be two nonadjacent vertices of  $G$  with degree sum at least  $n - 1$ . Note that a vertex is nonadjacent to itself. Hence  $x, y$  have at least one common neighbor.

Firstly we assume that  $x, y$  have at least three common neighbors, say  $z, z', z''$ . Since  $G$  is claw-free, either  $zz'$  or  $zz''$  or  $z'z''$  is in  $E(G)$ . Without loss of generality, we assume that  $zz' \in E(G)$ . Then  $z$  is a bad vertex, a contradiction.

Secondly we assume that  $x, y$  have two common neighbors, say  $z, z'$ . If  $zz' \in E(G)$ , then  $z$  will be a bad vertex. So we have that  $zz' \notin E(G)$ . Let  $C_x, C'_x, C_y, C'_y$  be the maximal cliques of  $G$  containing  $\{x, z\}, \{x, z'\}, \{y, z\}, \{y, z'\}$ , respectively. Clearly  $H = G[C_x \cup C'_x \cup C_y \cup C'_y]$  has a Hamilton cycle. Note that there is at most one vertex in  $V(G) \setminus V(H)$ . Since  $G$  is connected, we have that  $G$  is traceable.

Finally we assume that  $x, y$  have only one common neighbor  $z$ . Then every vertex is adjacent either to  $x$  or to  $y$ . This implies that  $G$  consists of at most four maximal cliques and  $G$  is a block-chain. Clearly in this case  $G$  is traceable.  $\square$

The following two lemmas are crucial in the proofs of our two theorems. We guess that they are of interest in their own rights.

**Lemma 2.** *Let  $G$  be a connected claw-free graph on  $n$  vertices and  $m$  edges. If*

$$m \geq \binom{n-3}{2} + 2,$$

*then  $G$  is traceable unless  $G = N_{n-3,3}$  or  $L$ .*

*Proof.* Let  $G' = cl(G)$  be the closure of  $G$ . Then

$$e(G') \geq m \geq \binom{n-3}{2} + 2.$$

If  $G'$  is  $N$ -free, then by Theorems 7 and 9,  $G'$  is traceable, and so is  $G$  by Theorem 10. Now we assume that  $G'$  contains an induced subgraph  $H \sim N$ . We denote the vertices of  $H$  as in Fig. 2. In the following part of this proof, we set  $N_H(x) = N_{G'}(x) \cap V(H)$  and  $d_H(x) = |N_H(x)|$ .

For any  $x \in V(G - H)$ , note that the neighborhood of  $x$  in  $G'$  is either a clique or the disjoint union of two cliques. But any at least four vertices of  $H$  do not form a clique or a disjoint union of two cliques. This implies that  $d_H(x) \leq 3$  for any  $x \in V(G - H)$ . Thus

$$e(G') = e(H) + e(G' - H) + e_{G'}(H, G' - H) \leq 6 + \binom{n-6}{2} + 3(n-6) = \binom{n-3}{2} + 3.$$

Recall that  $e(G') \geq \binom{n-3}{2} + 2$ . Thus we have  $e(G') = \binom{n-3}{2} + 2$  or  $\binom{n-3}{2} + 3$ .

**Case 1.**  $e(G') = \binom{n-3}{2} + 3$ .

In this case,  $G' - H$  is complete and every vertex in  $G' - H$  has exactly three neighbors in  $H$ . Suppose first that there is a vertex  $x$  in  $G' - H$  such that  $N_H(x) = \{a_1, a_2, a_3\}$ . We claim for every vertex  $x'$  in  $G' - H$ ,  $N_H(x') = \{a_1, a_2, a_3\}$ . Since  $N_H(x') \neq \{b_1, b_2, b_3\}$ , we assume without loss of generality that  $a_1 \in N_H(x')$ . Note that  $xx' \in E(G)$  and  $G'[N_{G'}(x)]$  is a clique or disjoint union of two cliques. We can see that  $a_2, a_3 \in N_H(x')$ . Hence as we claimed  $N_H(x') = \{a_1, a_2, a_3\}$ . Thus  $G' = N_{n-3,3}$ .

Suppose that  $E(G') \setminus E(G) \neq \emptyset$ . Then  $e(G) = \binom{n-3}{2} + 2$  and there is only one edge  $e$  in  $E(G') \setminus E(G)$ . If  $e$  is a pendant edge, then  $G$  is disconnected, a contradiction. So we assume that  $e = uv$  is not a pendant edge. Suppose without loss of generality that  $a_1$  is a vertex in  $\{a_1, a_2, a_3\} \setminus \{u, v\}$ . Then the subgraph induced by  $\{a_1, b_1, u, v\}$  is a claw in  $G$ , a contradiction. This implies that  $E(G') \setminus E(G) = \emptyset$ . Hence  $G = G' = N_{n-3,3}$ .

Now we assume that for every vertex  $x \in V(G - H)$ ,  $N_H(x) \neq \{a_1, a_2, a_3\}$ .

If  $V(G' - H) = \emptyset$ , then  $G' = N = N_{3,3}$ . By the analysis above, we can also see that  $G = G' = N_{3,3}$ . So we assume that  $V(G' - H) \neq \emptyset$ .

Let  $x$  be a vertex in  $G' - H$ . Thus  $N_H(x)$ , and then  $N(x)$  induces two disjoint cliques. Note that  $N_H(x) \neq \{b_1, b_2, b_3\}$ . We assume without loss of generality that  $a_1 \in N_H(x)$ . If  $a_2 \in N_H(x)$ , then  $a_3 \in N_H(x)$ ; otherwise  $a_1$  will be a bad vertex of  $G'$ . But in this case  $N_H(x) = \{a_1, a_2, a_3\}$ , a contradiction. This implies that  $a_2 \notin N_H(x)$  and similarly,  $a_3 \notin N_H(x)$ . Note that  $N_H(x) \neq \{a_1, b_2, b_3\}$ . We have  $b_1 \in N_H(x)$ . Without loss of generality, we assume that  $N_H(x) = \{a_1, b_1, b_2\}$ . If  $G' - H$  has the only one vertex  $x$ , then  $b_1 a_1 x b_2 a_2 a_3 b_3$  is a Hamilton path of  $G'$ . By Theorem 10,  $G$  is traceable. Now we assume that there is a second vertex  $x' \in V(G' - H)$ .

Since both  $\{x, x', b_1, b_2\}$  and  $\{x, x', b_2, a_2\}$  induce no claws, it follows either  $a_1, b_1 \in N_H(x')$  or  $b_2 \in N_H(x')$ . If  $a_1, b_1 \in N_H(x')$ , then  $b_2 \notin N_H(x')$ ; otherwise  $x$  is a bad vertex of  $G'$ . Similarly as the case of  $x$  above, we can see that  $a_2, a_3 \notin N_H(x')$ . Thus  $N_H(x') = \{a_1, b_1, b_3\}$ . If  $b_2 \in N_H(x')$ , then  $a_1, b_1 \notin N_H(x')$ ; otherwise  $x$  is a bad vertex of  $G'$ . If  $a_2 \in N_H(x')$ , then  $b_2$  is a bad vertex of  $G'$ , a contradiction. Thus we have  $N_H(x') = \{b_2, a_3, b_3\}$ . In conclusion, either  $N_H(x') = \{a_1, b_1, b_3\}$  or  $N_H(x') = \{b_2, a_3, b_3\}$ .

Suppose that there is a third vertex  $x''$ . Then similarly as the case of  $x'$ ,  $N_H(x'') = \{a_1, b_1, b_3\}$  or  $N_H(x'') = \{b_2, a_3, b_3\}$ . But if  $x'$  and  $x''$  have the same neighborhood in  $H$ , then  $x'$  will be a bad vertex, a contradiction. So we assume without loss of generality that  $N_H(x') = \{a_1, b_1, b_3\}$  and  $N_H(x'') = \{b_2, a_3, b_3\}$ . Then  $x'$  is also a bad vertex, a contradiction. Thus  $x, x'$  are the only two vertices in  $G - H$ , and  $b_1 x x' b_3 a_3 a_1 a_2 b_2$  is a Hamilton path of  $G'$ . By Theorem 10,  $G$  is traceable.

**Case 2.**  $e(G') = \binom{n-3}{2} + 2$ .

In this case  $G = G'$  and there is a vertex  $x$  in  $G - H$  such that  $d_H(x) = 2$  or  $xx' \notin E(G)$  for some  $x' \in V(G - H)$ . Let  $G_1 = G - x$ . Since every vertex in  $G - H - x$  is adjacent to three vertices in  $H$ ,  $G_1$  is connected. Note that

$$e(G_1) = e(G) - d(x) = \binom{n-3}{2} + 2 - (n-5) = \binom{n-4}{2} + 3.$$

Using the conclusion of Case 1, we can obtain that  $G_1$  is traceable or  $G_1 = N_{n-4,3}$ .

Suppose first that  $G_1 = N_{n-4,3}$ . Let  $a_1b_1, a_2b_2, a_3b_3$  be the three pendent edges of  $G_1$ , where  $a_1, a_2, a_3$  are contained in a clique of  $G_1$ . Note that  $G$  is closed and  $N(x)$  is either a clique or the disjoint union of two cliques. Also note that if  $x$  is adjacent to some two vertices of a maximal clique of  $G$ , then  $x$  will be adjacent to every vertex of the maximal clique of  $G$ . Since  $d(x) = n - 5$ , the neighborhood of  $x$  does not include  $V(G_1) \setminus \{b_1, b_2, b_3\}$ . If  $x$  is adjacent to two pendant vertices, say  $b_1, b_2$ , then let  $P$  be a Hamilton path of the complete graph  $G_1 - \{b_1, b_2, b_3\}$  from  $a_2$  to  $a_3$ . Then  $b_1xb_2a_2Pa_3b_3$  is a Hamilton path of  $G$ . Now we assume that  $x$  is adjacent to exactly one vertex of  $\{b_1, b_2, b_3\}$ . Suppose without loss of generality that  $b_1 \in N(x)$ . Since  $d(x) = n - 5$ , we can see that  $n = 7$ ,  $G_1 = N$  and  $N(x) = \{a_1, b_1\}$ . Hence  $G = L$ .

Now we assume that  $G_1$  is traceable. Let  $P = v_1v_2 \dots v_{n-1}$  be a Hamilton path of  $G_1$ . If  $v_1x \in E(G)$  or  $v_{n-1}x \in E(G)$ , then  $G$  is traceable. So we assume that  $v_1x, v_{n-1}x \notin E(G)$ . If  $x$  is adjacent to two successive vertices on  $P$ , then  $G$  is traceable. So we assume that  $x$  is not adjacent to two successive vertices on  $P$ . This implies that  $n - 1 - d(x) \geq d(x) + 1$ . Since  $d(x) = n - 5$ , we have  $n \leq 8$ . Note that  $n \geq 7$ . We can see that either  $xv_2$  or  $xv_{n-2}$  is in  $E(G)$ . We assume without loss of generality that  $xv_2 \in E(G)$ . Thus  $v_1v_3 \in E(G)$ ; otherwise the subgraph induced by  $\{v_2, v_1, v_3, x\}$  is a claw. Hence  $P' = xv_2v_1v_3 \dots v_{n-1}$  is a Hamilton path of  $G$ .  $\square$

**Lemma 3.** *Let  $G$  be a connected claw-free graph on  $n \geq 24$  vertices and  $m$  edges. If*

$$m > \binom{n}{2} - (1 + \sqrt{3n-8})^2,$$

*then  $G$  is traceable unless  $G \subseteq N_{n-3,3}$ .*

*Proof.* We assume the opposite.

**Claim 1.**  $G$  is a block-chain.

*Proof.* Suppose that  $G$  is not a block-chain. Since  $G$  is claw-free, every cut-vertex of  $G$  is contained in exactly two blocks. This implies that  $G$  has a block  $B_0$  which contains at least three cut-vertices of  $G$ . Let  $a_1, a_2, a_3$  be three cut-vertices of  $G$  contained in  $B_0$ . Let  $B_i$ ,  $i = 1, 2, 3$ , be the component of  $G - B_0$  which has a neighbor of  $a_i$ . Let

$H_0 = G - (\bigcup_{i=1}^3 B_i)$  and  $H_i = G[V(B_i) \cup \{a_i\}]$ ,  $i = 1, 2, 3$ . Note that  $\nu(H_0) \geq 3$ . If  $\nu(H_1) = \nu(H_2) = \nu(H_3) = 2$ , then  $G \subseteq N_{n-3,3}$ . Now we assume without loss of generality that  $\nu(H_1) \geq 3$ .

Note that  $\sum_{i=0}^3 \nu(H_i) = n + 3$ . Thus

$$e(G) = \sum_{i=0}^3 e(H_i) \leq \sum_{i=0}^3 \binom{\nu(H_i)}{2} \leq \binom{n-4}{2} + 5 \leq \binom{n}{2} - (1 + \sqrt{3n-8})^2$$

(noting that  $n \geq 24$ ), a contradiction.  $\square$

Let  $G' = cl(G)$ . If  $G'$  is  $M$ -free, then by Theorems 8 and 10,  $G'$ , and then  $G$ , is traceable. Now we assume that  $G'$  has an induced subgraph  $H \sim M$ . We denote the vertices of  $H$  as in Fig. 2.

**Claim 2.** Every vertex in  $G' - H$  has at most 5 neighbors in  $H$ ; and there is at most one vertex in  $G' - H$  having exactly 5 neighbors in  $H$ .

*Proof.* Let  $x$  be a vertex in  $G' - H$ . Note that  $N_H(x)$  is either a clique or the disjoint union of two cliques. This implies that  $d_H(x) \leq 5$ . Moreover, if  $d_H(x) = 5$ , then  $N_H(x) = \{a_1, a_2, a_3, c_1, d_1\}$ .

If there are two vertices, say  $x$  and  $x'$ , such that each one has 5 neighbors in  $H$ , then  $N_H(x) = N_H(x') = \{a_1, a_2, a_3, c_1, d_1\}$ . But in this case  $x$  will be a bad vertex of  $G'$ , a contradiction.  $\square$

By Claim 2, we have

$$e(G) \leq e(G') = e(H) + e(G' - H) + e_{G'}(H, G' - H) \leq 8 + \binom{n-8}{2} + 4(n-8) + 1.$$

Thus

$$8 + \binom{n-8}{2} + 4(n-8) + 1 > \binom{n}{2} - (1 + \sqrt{3n-8})^2.$$

This implies that  $n \leq 20$ , a contradiction.  $\square$

The next theorem we need is a famous theorem due to Hong [12]. In fact, the spectral inequality also works for graphs without isolated vertices, see [12].

**Theorem 11** (Hong [12]). *Let  $G$  be a connected graph on  $n$  vertices and  $m$  edges. Then*

$$\mu(G) \leq \sqrt{2m - n + 1}.$$

*The equality holds if and only if  $G = K_n$  or  $K_{1,n-1}$ .*

**Theorem 12** (Hofmeister [11]). *Let  $G$  be a graph. Then*

$$\mu(G) \geq \sqrt{\frac{\sum_{v \in V(G)} d^2(v)}{n}}.$$



### 3 Proofs of the main results

**Proof of Theorem 5.** By Theorem 11,  $\mu(G) \leq \sqrt{2m - n + 1}$ . Thus  $n - 4 \leq \sqrt{2m - n + 1}$  and

$$m \geq \left\lceil \frac{(n-3)(n-4) + 3}{2} \right\rceil = \binom{n-3}{2} + 2.$$

Note that  $\mu(M) = 2.6935 \dots < 3$ . By Lemma 2,  $G$  is traceable or  $G = N_{n-3,3}$ .  $\square$

**Proof of Theorem 6.** We first give a bound on the value of  $\mu(\overline{N_{n-3,3}})$ . By using Theorem 2.8 in [4] and some computing, we know

$$\mu(K_k \vee (n-k)K_1) = \frac{k-1 + \sqrt{4kn - (3k-1)(k+1)}}{2}.$$

Thus  $\mu(K_3 \vee (n-3)K_1) = 1 + \sqrt{3n-8}$ . From the fact  $\overline{N_{n-3,3}} \subset K_3 \vee (n-3)K_1$ , we obtain

$$\mu(\overline{N_{n-3,3}}) < 1 + \sqrt{3n-8}$$

for any  $n \geq 6$ .

Now we prove the theorem. The idea of our proof comes from [8]. We assume that  $G$  is not traceable. Let  $G' = cl(G)$ . By Theorem 10,  $G'$  is not traceable. By Lemma 1, for any pair of nonadjacent vertices  $u, v$  of  $G'$ ,  $d_{G'}(u) + d_{G'}(v) \leq n-2$ , and hence

$$d_{\overline{G'}}(u) + d_{\overline{G'}}(v) \geq 2(n-1) - (n-2) = n.$$

Furthermore, we have

$$\sum_{v \in V(G)} d_{\overline{G'}}^2(v) = \sum_{uv \in E(\overline{G'})} (d_{\overline{G'}}(u) + d_{\overline{G'}}(v)) \geq ne(\overline{G'}).$$

Note that  $\overline{G'} \subseteq \overline{G}$ . By Theorem 12,

$$\mu(\overline{G}) \geq \mu(\overline{G'}) \geq \sqrt{\frac{\sum_{v \in V(G)} d_{\overline{G'}}^2(v)}{n}} \geq \sqrt{e(\overline{G'})}.$$

Thus we have

$$e(G') = \binom{n}{2} - e(\overline{G'}) \geq \binom{n}{2} - \mu^2(\overline{G}) > \binom{n}{2} - (1 + \sqrt{3n-8})^2.$$

Recall that  $G'$  is claw-free and not traceable. By Lemma 3,  $G' \subseteq N_{n-3,3}$ . Thus  $G \subseteq N_{n-3,3}$ . But if  $G \subset N_{n-3,3}$ , then  $\mu(\overline{G}) > \mu(\overline{N_{n-3,3}})$ , a contradiction. This implies  $G = N_{n-3,3}$ . The proof is complete.  $\square$

## 4 Concluding remarks

In this section, we give a brief discussion of the existence of Hamilton cycles in claw-free graphs under spectral condition.

Following the notations in [2], we use  $\mathcal{P}$  to denote the class of graphs obtained by taking two vertex-disjoint triangles  $a_1a_2a_3a_1$  and  $b_1b_2b_3b_1$ , and by joining every pair of vertices  $\{a_i, b_i\}$  by a triangle or by a path of order at least 3. We use  $P_{x_i, x_2, x_3}$  to denote the graph from  $\mathcal{P}$ , where  $x_i = T$  if  $\{a_i, b_i\}$  is joined by a triangle; and  $x_i = k_i$  if  $\{a_i, b_i\}$  is joined by a path of order  $k_i \geq 3$ .

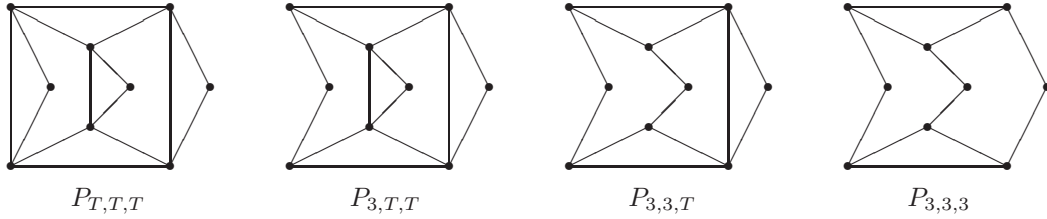


Fig. 3. 2-connected claw-free non-Hamiltonian graphs of order 9.

Brousek [2] showed that every 2-connected claw-free non-Hamiltonian graph contains a graph in  $\mathcal{P}$  as an induced subgraph. By Brousek's result, we can see that the smallest 2-connected claw-free non-Hamiltonian graphs have order 9, and there are exactly four such graphs, namely,  $P_{T,T,T}$ ,  $P_{3,T,T}$ ,  $P_{3,3,T}$  and  $P_{3,3,3}$ , see Fig. 3.

Let  $H$  be a graph from Fig. 3, and let  $G$  be a graph obtained from  $H$  by replacing one triangle by a complete graph  $K_{n-6}$ . Then  $G$  is not Hamiltonian and  $\mu(G) > n - 7$ . Recently, we get the following result.

**Theorem 13.** *Suppose that  $G$  is a 2-connected claw-free graph of sufficiently large order  $n$ . If  $\mu(G) \geq n - 7$ , then  $G$  is Hamiltonian or  $G$  is a subgraph of a graph which is obtained from  $P_{T,T,T}$ ,  $P_{3,T,T}$ ,  $P_{3,3,T}$  or  $P_{3,3,3}$  by replacing a triangle by  $K_{n-6}$ .*

For further works on this topic, we refer the reader to [14].

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